

A systematic approach to model building

Damien P. George*

*Nikhef Theory Group, Science Park 105,
1098 XG Amsterdam, The Netherlands*

(Dated: January 25, 2013)

Abstract

We outline a new, systematic way of constructing and analysing field theories, where all possible continuous symmetries of a given model are derived using the method of Lie point symmetries. If the model has free parameters, and relationships amongst any of these parameters yields an enhanced symmetry, then all such relationships are found, along with the resulting symmetry group. We discuss how the method can be applied to the standard model and beyond, to direct the search for a more predictive field theory. The method handles compact and non-compact continuous groups, spontaneously broken symmetries, and is also applicable to general relativity.

*Electronic address: dpgeorge@nikhef.nl

I. INTRODUCTION

If one wanted to compute all the possible symmetries of a Lagrangian, one could naively write all coordinates and fields as arbitrary functions of every coordinate and field, demand that the resulting Lagrangian was the same as the original, and then solve for the arbitrary functions. As long as one can obtain all the solutions, this method would give an *exhaustive* list of the symmetries of the model. Furthermore, if there are free parameters in the model that, when given specific values or relationships to other parameters, yield a different set of symmetries, then the method would necessarily cover these cases as well.

The obvious problem with such a method is that one would obtain equations which are just as, if not more, difficult to solve than the entire system itself. But the general idea is a powerful one and we would like to implement it in at least some reduced form. In fact, if we are content with finding only continuous symmetries then, by the virtue that a continuous group is classified by its local Lie algebra, the arbitrary functions mentioned above need only be infinitesimal and the resulting set of equations that need to be solved are linear partial differential equations (PDEs).

Such a technique exists and was originally worked out by Lie in order to assist in solving differential equations. If one knows a certain solution of a set of PDEs, then knowing the symmetries allows one to easily construct other solutions. This was the motivation behind Lie's development of the theory. The method is now known by various names depending on how general a symmetry one is looking for. Here we shall be concerned with the Lie point symmetry (LPS), which is a continuous symmetry that can depend on coordinates and fields, but not on derivatives of the fields. The LPS method involves finding and solving a set of partial differential equations known as the determining equations. These "determine" the allowed symmetries, and, when solved in a systematic way, yield all interesting relationships between free parameters of the original system.

The basic text book is by Olver [1] and discusses in depth most of the theory related to finding symmetries. A useful algorithm for solving a large set of PDEs is described in detail by Reid [2], and further improvements on this are given in Refs. [3, 4]. The idea of using the LPS method to find all the symmetries of a field theory has previously been applied to the case of scalar QED and Weyl QED [5], to Einstein's equations [6], and to Yang-Mills in 4d [7]. The extension to supersymmetry has also been developed and utilised [8]. A

comprehensive survey of this field is given in [9], with an emphasis on available computer programs to automatically carry out the LPS method. These past studies have overlooked the utility of the method for particle physics and model building, in particular the ability to find parameter relationships that yield a larger symmetry group.

In this paper we shall show in detail how the LPS method works and discuss its application to field theories. Using these ideas one has a new approach to model building, where symmetries and parameter relationships are systematically *derived*, not input from the start. We begin in Section II with a description of the LPS method, followed by a simple example in Section III. In Section IV we derive the allowed symmetries of a general theory with N interacting scalars, and specialise to cases with low N . Section V discusses, with the aid of a worked example, the difference between the symmetries of the action and those of the equations of motion. We make some remarks on the utility of the LPS method in Section VI, including its ability to handle spontaneously broken symmetries. A way to automate the method is outlined in Section VII, along with a plan for the construction of a catalogue of field theories. The ultimate aim is to be able to apply the LPS method to the standard model, and beyond, to give a more directed search to new theories of particle physics, a topic which is discussed in Section VIII. We conclude the paper in Section IX.

II. THE LIE POINT SYMMETRY METHOD

For a given system, the Lie point symmetry method consists of finding the associated determining equations, whose solutions describe infinitesimal symmetries, and then solving these equations. The term “point” means that the finite transformations of the coordinates and fields depend only on the coordinates and fields themselves, and not the derivatives of the fields. According to Lie, one does not have to look at the full finite transformation; it is enough to study their infinitesimal behaviour. The method proceeds as follows.

1. Derive the determining equations of the system.

Given an action, or equations of motion, with coordinates x^μ and fields ϕ_i , one makes a general infinitesimal variation of the coordinates, $\delta x^\mu = \eta^\mu$, and fields, $\delta \phi_i = \chi_i$, and obtains equations — the so-called determining equations — which are linear partial differential equations for η^μ and χ_i . The solutions of these determining equations describe the set of symmetries of the original system.

2. Solve the determining equations, or at least reduce them to a standard form.

This is an involved step. For simple systems the determining equations are only weakly coupled and can be completely solved, or reduced to algebraic equations, with little effort. This is not the case for more complicated systems. However, due to the fact that the determining equations are linear, there is a well-defined algorithm which completes in finite time and brings the determining equations to a standard form, where they are in involution. The algorithm is in essence Gaussian elimination: an ordering on η^μ and χ_i is defined, the terms of the equations are sorted using this ordering, each equation is treated as a row, and the system is reduced to “diagonal” form.

A critical part of the reduction is the “column elimination” step. The coefficient of the leading term used in the elimination is in general a function of the free parameters of the original model. If the values of the parameters have specific values such that this coefficient is zero, this particular elimination cannot proceed. The algorithm must then branch, with one branch of the solution corresponding to the coefficient being zero, and the other branch, being non-zero. The algorithm then continues with each branch independently, possibly spawning additional branches as the reduction proceeds. In general, the resulting solution of each branch, and hence the symmetries, are different. This branching is a general feature of the LPS method. Given a model, for each set of relationships among its parameters that yield a different symmetry group the LPS method will produce a branch associated with this set, and the parameter relationship will be specified.

3. Compute the rank of the symmetry set(s).

For each branch we compute the amount of symmetry — the rank — by counting the number of integration constants (the initial data) in the reduced set of determining equations. If the equations of the branch are solved then one has already found these integration constants. If the branch is not solved, but is in standard reduced form, one can unambiguously compute the amount of initial data needed to fully specify a unique solution to the set of differential equations. Either way, the amount of initial data gives the rank of the symmetry corresponding to the solutions of that branch.

The notation for the rank is a tuple of integers $R = (N_{\text{const}}, N_{f_1}, N_{f_2} \dots)$. Here, N_{const}

is the number of independent constants, N_{f_1} the number of independent functions of one variable, N_{f_2} the number of functions of two variables, and so on. Trailing zeros in the tuple will generally be suppressed. Each constant and function parameterises an independent symmetry, so the rank gives a compact and precise indication of the amount of symmetry in a given system.

4. Compute the action of the symmetries.

This last step is not always possible, and only necessary if one wants to know how a certain symmetry acts on the coordinates and fields. One must solve explicitly for the forms of η^μ and χ_i , if they have not already been obtained, and then compute the action of the symmetry on the coordinates and fields. One uses the usual techniques for solving differential equations to do this.

As we shall demonstrate, the LPS method is a very general and powerful tool, and allows one to build and analyse field theories in a completely systematic way. We emphasise that the method

- is an exhaustive search of continuous symmetries,
- yields all interesting relationships between parameters, and
- is guaranteed to terminate (up to the end of step 3) in finite time, determined by the number of coordinates and number of fields.

Of course, it has some drawbacks. There is no guarantee that one can solve for the actions of the symmetries (step 4), although it is possible to solve for the structure constants of the group [4]. Apart from this, the biggest disadvantage of the LPS method is that the number and complexity of determining equations increases rapidly (but still polynomially) with the number of fields ϕ . The number of equations also increases drastically with the number of coordinates x , but this is not such a problem if one sticks with 4d theories.¹ The growth with number of fields is our major concern, and it seems that the only way forward is to automate the above 3 (or 4) steps. Actually, these steps lend themselves quite nicely to automation, as we shall discuss in Section VII.

¹ If one can show, as we do for a specific set of theories in Section IV, that the symmetries of the coordinate and field sectors separate, then having a large number of dimensions is no problem.

We now describe in more detail step 1: how to derive the set of determining equations for a given system. One can start from the action, or from the equations of motion.

A. The action approach

Given an action, one makes a general variation by adding infinitesimals to all the coordinates and fields. Demanding that the result is equivalent to the original action gives the master determining equation, from which one obtains the individual set of determining equations. We shall derive the form of the master equation for a general Lagrangian that depends only on coordinates, fields and first derivatives of fields. For more detail see [1, 10].

Consider then a general action of real fields $\phi_i(x)$ (which could be the components of a field with arbitrary spin properties) in arbitrary dimensions:

$$\mathcal{S} = \int d^n x \mathcal{L}[x^\mu, \phi_i(x), \partial_\mu \phi_i(x)] . \quad (1)$$

The infinitesimal point transformation is

$$x^\mu \rightarrow \bar{x}^\mu = x^\mu + \eta^\mu(x, \phi) , \quad \phi_i \rightarrow \bar{\phi}_i = \phi_i + \chi_i(x, \phi) . \quad (2)$$

Under this transformation one can show that the action transforms to

$$\mathcal{S} \rightarrow \int d^n x \left[\mathcal{L} + \mathcal{L} \frac{d\eta^\mu}{dx^\mu} + \frac{\partial \mathcal{L}}{\partial x^\mu} \eta^\mu + \frac{\partial \mathcal{L}}{\partial \phi_i} \chi_i + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i)} \left(\frac{d\chi_i}{dx^\mu} - \frac{\partial \phi_i}{\partial x^\nu} \frac{d\eta^\nu}{dx^\mu} \right) \right] . \quad (3)$$

Sum over repeated μ, ν and i indices is understood. The total derivative is

$$\frac{d}{dx^\mu} \equiv \frac{\partial}{\partial x^\mu} + \frac{\partial \phi_i}{\partial x^\mu} \frac{\partial}{\partial \phi_i} . \quad (4)$$

In deriving this equation we have used the Jacobian transformation matrix for the coordinates, $J = \partial \bar{x} / \partial x = \mathbb{1} + \partial \eta / \partial x$, which transforms x to \bar{x} . Some useful properties are $J^{-1} = \mathbb{1} - \partial \eta / \partial x$ and $\det(J) = 1 + \text{Tr}(\partial \eta / \partial x)$.²

The infinitesimals η and χ do not necessarily vanish on the boundary (for example, a time translation symmetry has η^t a constant) so we cannot do integration by parts on (3).

² Note that if we take equation (3), set $\eta = 0$, and do integration by parts on one of the remaining terms, then we can obtain the usual Euler-Lagrange equations for the ϕ_i .

This equation is thus in its simplest form, and the condition for the transformation to leave the action invariant is³

$$\mathcal{L} \frac{d\eta^\mu}{dx^\mu} + \frac{\partial \mathcal{L}}{\partial x^\mu} \eta^\mu + \frac{\partial \mathcal{L}}{\partial \phi_i} \chi_i + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i)} \left(\frac{d\chi_i}{dx^\mu} - \frac{\partial \phi_i}{\partial x^\nu} \frac{d\eta^\nu}{dx^\mu} \right) = 0. \quad (5)$$

This is a key result. It is the master determining equation. For a given Lagrangian density one computes the above expression explicitly, treats all the *derivatives* of the fields ϕ_i as independent variables, and then equates the independent coefficients to zero. The resulting equations are the determining equations. Note that the result can be extended in a straightforward way to the case where the Lagrangian density depends on higher order derivatives of the fields.

Once η^μ and χ_i have been solved for, one can obtain the finite action of the symmetry group by solving the coupled differential equations

$$\frac{d\bar{x}^\mu}{d\epsilon} = \eta^\mu(\bar{x}, \bar{\phi}), \quad \frac{d\bar{\phi}_i}{d\epsilon} = \chi_i(\bar{x}, \bar{\phi}), \quad (6)$$

where ϵ is the continuous group parameter. The initial conditions for this set of equations are $\bar{x}^\mu(0) = x^\mu$ and $\bar{\phi}_i(0) = \phi_i$.

In summary, the idea is the following. Given an action, one varies it by adding infinitesimals to the variables and demanding that the resulting action is the same as the original. In the usual case, one lets the *infinitesimals be arbitrary and solves for the field configuration*. This yields the Euler-Lagrange equations which are classical solutions that extremise the action. The situation here is the reverse: in equation (5) we let the *field derivatives be arbitrary and solve for the infinitesimals*. This gives the determining equations, which are a set of coupled *linear and first order* PDEs.⁴ In some sense we have, in this way, obtained a linear version of the theory, whose solution gives all the symmetries of the action. And, as we have pointed out before, when finding the symmetries all the interesting relationships between parameters of the model become apparent in a systematic way.

³ One could equate this to a total coordinate derivative to give more freedom, and possibly additional symmetries. We do not consider such a case here.

⁴ The determining equations coming from the Euler-Lagrange equations are in general second order because equations of motion have second derivatives of the fields in them. Here the determining equations are first order because \mathcal{L} (by assumption) has only first derivatives in it.

B. The equations of motion approach

It is also possible to obtain the determining equations from the equations of motion of the system, the Euler-Lagrange equations. This approach requires more effort than the action approach since an action with only first-order derivatives of the fields will in general have second-order equations of motion. Nevertheless, it is worth discussing this alternative technique since it is suited to finding continuous symmetries of any set of PDEs, not just those of a field theory.

The system under consideration can be a function of both independent, x^μ , and dependent, $\phi_i(x)$, variables, and can include derivatives of the ϕ_i . Denote the system by

$$\Delta_j(x^\mu, \phi_i, \partial\phi_i) = 0, \quad (7)$$

where j indexes each equation of the system and $\partial\phi_i$ can be a derivative of arbitrary order. Given this system of PDEs, the determining equations are obtained as follows.

1. Construct the prolonged symmetry operator $\text{pr}^{(k)} \alpha$.

Point symmetries of the system described by Δ take the form $x^\mu \rightarrow \bar{x}^\mu = X^\mu(x, \phi)$, $\phi_i \rightarrow \bar{\phi}_i = \Phi_i(x, \phi)$, with $\Delta_j(X_i, \Phi_i, \partial\Phi_i) = 0$. We construct the differential operator α which, when applied to an object, realises the infinitesimal point symmetry transformation:

$$\alpha = \eta^\mu \frac{\partial}{\partial x^\mu} + \chi_i \frac{\partial}{\partial \phi_i}, \quad (8)$$

with implicit sum over the indices. Because the system Δ can depend on derivatives of ϕ , the operator α must be prolonged to $\text{pr}^{(k)} \alpha$ so that it acts correctly on this extended space of functions. k here is the highest order derivative of the system. The formula for $\text{pr}^{(k)}$ is complicated and will not be given here; see [1] or [9]. It essentially extends α to include all possible combinations of derivatives of ϕ , to order k .

2. Apply $\text{pr}^{(k)} \alpha$ to the system.

We demand the following holds:

$$(\text{pr}^{(k)} \alpha \cdot \Delta)|_{\Delta=0} = 0. \quad (9)$$

This means that the action of the infinitesimal symmetry generator leaves the system unchanged, when evaluated on a solution.

3. Obtain the determining equations by equating all independent coefficients to zero.

The independent variables in equation (9) are derivatives $\partial\phi$ (since η and χ only depend on x and ϕ). Thus (9) holds in general only when all the coefficients of these derivatives vanish. Extracting these coefficients and setting them to zero yields the determining equations. These equations are a set of linear PDEs in the variables η and χ , which both depend on x and ϕ .

A more detailed description of these steps can be found in, for example, [1] or [9]. We shall give an example using the equations of motion approach in Section V, and also discuss its differences to the action approach.

III. AN EXAMPLE: TWO SCALARS AND A U(1) SYMMETRY

The best way to understand the LPS method is to work through an example. Let us do that by finding the symmetries of two massive scalars (spin-0 fields). We shall ignore the coordinate sector to keep things simple. This still allows us to see the workings of the method and an example of branching for different values of parameters, in this case the masses.

The Lagrangian density in an arbitrary number of dimensions is

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi_1\partial_\mu\phi_1 + \frac{1}{2}\partial^\mu\phi_2\partial_\mu\phi_2 - \frac{1}{2}m_1^2\phi_1^2 - \frac{1}{2}m_2^2\phi_2^2. \quad (10)$$

There are two independent field variables and we must solve for their variations, $\chi_1(\phi_1, \phi_2)$ and $\chi_2(\phi_1, \phi_2)$. The master determining equation is found by feeding the above Lagrangian density into equation (5). One obtains

$$-m_1^2\phi_1\chi_1 - m_2^2\phi_2\chi_2 + \partial^\mu\phi_1\partial_\mu\phi_1\frac{\partial\chi_1}{\partial\phi_1} + \partial^\mu\phi_1\partial_\mu\phi_2\frac{\partial\chi_1}{\partial\phi_2} + \partial^\mu\phi_2\partial_\mu\phi_1\frac{\partial\chi_2}{\partial\phi_1} + \partial^\mu\phi_2\partial_\mu\phi_2\frac{\partial\chi_2}{\partial\phi_2} = 0. \quad (11)$$

Since the χ 's do not depend on derivatives of the fields we must have the coefficients of all independent derivative factors vanish. This gives the four determining equations for our system:

$$\begin{aligned} \frac{\partial\chi_1}{\partial\phi_1} &= 0, & \frac{\partial\chi_1}{\partial\phi_2} + \frac{\partial\chi_2}{\partial\phi_1} &= 0, & \frac{\partial\chi_2}{\partial\phi_2} &= 0, \\ -m_1^2\phi_1\chi_1 - m_2^2\phi_2\chi_2 &= 0. \end{aligned} \quad (12)$$

These equations are simple enough that we can solve them directly. The first three equations give the general solution

$$\chi_1(\phi_2) = \alpha_1 + \beta\phi_2, \quad \chi_2(\phi_1) = \alpha_2 - \beta\phi_1, \quad (13)$$

with $\alpha_{1,2}$ and β constants. As we shall see, each of these correspond to a specific symmetry. The maximum rank of the system is $R = (3)$ because we have three independent constants, but it can be less for various values of the parameters m_i^2 . Substituting the solutions (13) in the remaining determining equation we find

$$\alpha_1 m_1^2 \phi_1 + \alpha_2 m_2^2 \phi_2 + \beta(m_1^2 - m_2^2) \phi_1 \phi_2 = 0. \quad (14)$$

This final determining equation is now in an algebraic form. It is a polynomial in the fields, and each coefficient of the polynomial must be independently zero.

Now comes the key observation: *the symmetries depend on the parameters*. If $m_1^2 = 0$ then α_1 is free. Using the definition (6), we read off the differential equations that this symmetry satisfies. Setting all other parameters to zero we obtain the differential equations $\bar{\phi}'_1(\epsilon) = \alpha_1$ and $\bar{\phi}'_2(\epsilon) = 0$. Recall that ϵ is the parameter of the continuous group. The solution is $\bar{\phi}_1(\epsilon) = \phi_1 + \alpha_1 \epsilon$ and $\bar{\phi}_2(\epsilon) = \phi_2$. We used the initial condition $\bar{\phi}_i(0) = \phi_i$ to fix the integration constants. Physically, this corresponds to a shift symmetry, and is only present when ϕ_1 is massless. Indeed, if $m_1^2 \neq 0$ then equation (14) is satisfied only when $\alpha_1 = 0$. Similarly, α_2 is the shift symmetry for ϕ_2 .

From equation (14) we can also see that a symmetry arises if $m_1^2 = m_2^2$. This symmetry has $\bar{\phi}'_1 = \beta\bar{\phi}_2$ and $\bar{\phi}'_2 = -\beta\bar{\phi}_1$. The solution is

$$\begin{pmatrix} \bar{\phi}_1 \\ \bar{\phi}_2 \end{pmatrix} = \begin{pmatrix} \cos \beta\epsilon & \sin \beta\epsilon \\ -\sin \beta\epsilon & \cos \beta\epsilon \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \quad (15)$$

This is a U(1) rotation symmetry, and is only present when the masses are equal. $(\phi_1, \phi_2) \rightarrow (\bar{\phi}_1, \bar{\phi}_2)$ leaves the action (and the Lagrangian in this case) invariant.

Each of the three independent symmetries we obtained has rank $R = (1)$ since they correspond to a single parameter. Depending on the values of the masses, the total rank of the system will be different. For example, if $m_1^2 = m_2^2 \neq 0$ then the total rank is $R = (1)$. If $m_1^2 = m_2^2 = 0$ then the total rank is $R = (3)$. It is important to note that the rank, and the specific parameter relationships leading to such a rank, can be computed *without* solving for

the actual finite action of the symmetries themselves. Furthermore, the rank and parameter relations can be obtained in a completely systematic way, as we shall describe in Section VII. This is a crucial point when analysing large systems.

That is the LPS method in a nutshell, from step 1, finding the determining equations, through step 4, solving for the action of the symmetries. It seems simple, but when applying it to large systems of equations one obtains orders of magnitude more determining equations, along with much more interesting structure in the symmetries.

IV. N INTERACTING SCALAR FIELDS

The symmetries of a theory are dictated by the structure of the interactions between fields, which are modelled by individual terms in the action. A generic theory with a certain set of interactions will allow for a certain set of symmetry groups. The precise symmetries are then fixed by the specific values of the coefficients of each interaction term. This was made clear in the previous section with the general form of the allowed set of symmetries given by equation (13), while the specific symmetries were dictated by the values of the two masses. In general it is the terms with *derivatives* in them that are most important in dictating the allowed set of symmetries. More complicated derivative terms allow for more sophisticated symmetries.

In this section we use the LPS method to perform a complete analysis of what is perhaps the simplest set of Lorentz invariant interactions: the action corresponding to N interacting spin-0 fields in D dimensions. The Lagrangian density is

$$\mathcal{L} = T_{ij} \partial^\mu \phi_i \partial_\mu \phi_j - V(\phi) , \quad (16)$$

where ϕ_i are the N fields with $i = 1 \dots N$, T_{ij} is a constant $N \times N$ symmetric matrix describing the kinetic mixing of the scalars, and $V(\phi)$ is an arbitrary potential of all N fields. We work in flat space with mostly minus signature, and a sum over repeated indices is implicit. We can diagonalise T_{ij} , and then, assuming all its eigenvalues are positive, rescale the fields to bring all the kinetic terms to canonical form; in effect $T_{ij} \rightarrow \frac{1}{2} \delta_{ij}$. This will change the form of V , but since it is arbitrary this makes no difference.

We now apply equation (5) to this general Lagrangian, in order to find the determining

equations. They are

$$V \partial_\mu \eta^\mu + \frac{\partial V}{\partial \phi_i} \chi_i = 0, \quad (17)$$

$$\partial^\mu \chi_i - V \frac{\partial \eta^\mu}{\partial \phi_i} = 0 \quad \forall \mu \forall i, \quad (18)$$

$$\partial^\mu \eta^\nu + \partial^\nu \eta^\mu = 0 \quad \forall \mu \forall \nu, \mu \neq \nu, \quad (19)$$

$$\frac{\partial \chi_i}{\partial \phi_j} + \frac{\partial \chi_j}{\partial \phi_i} = 0 \quad \forall i \forall j, i \neq j, \quad (20)$$

$$\frac{1}{2} \partial_\sigma \eta^\sigma - \partial_{\bar{\mu}} \eta^{\bar{\mu}} + \frac{\partial \chi_{\bar{i}}}{\partial \phi_{\bar{i}}} = 0 \quad \forall \bar{\mu} \forall \bar{i}, \quad (21)$$

$$\frac{\partial \eta^\mu}{\partial \phi_i} = 0 \quad \forall \mu \forall i. \quad (22)$$

Here a bar over an index indicates that an implicit sum should not be taken. Using equation (22) in equation (18) we find that $\partial^\mu \chi_i = 0$. Thus, the coordinate transformations η^μ do not depend on the fields, and the field transformations χ_i do not depend on the coordinates. This greatly simplifies the problem of solving the above set of equations.

The general solution for χ_i is

$$\chi_i(\phi) = \alpha_i + \beta_{ij} \phi_j + \gamma \phi_i, \quad (23)$$

where α_i and γ are constants, and β_{ij} is antisymmetric: $\beta_{ij} = -\beta_{ji}$. In terms of symmetries, α_i corresponds to a constant shift of the fields, β_{ij} to rotations among the fields, and γ to a communal scaling symmetry. These are, in general, the only symmetries allowed in the field sector for the generic Lagrangian (16), irrespective of the form of V . We shall see later though that the constants in equation (23) are dictated by the choice of V (or vice versa).

Let us now solve for η^μ . If we take equation (21), substitute in the solution for χ_i and sum over $\bar{\mu}$ we obtain

$$\left(\frac{1}{2} D - 1 \right) \partial_\sigma \eta^\sigma + D \gamma = 0. \quad (24)$$

At this point we get a branch in the solution, one for $D = 2$ and one for $D \neq 2$. Physically, this corresponds to the fact that the scaling symmetry behaves differently in two dimensions because the scalar field is dimensionless.

For the $D = 2$ branch we have $\gamma = 0$ and the general solution for η^μ is

$$\eta^t = F_+(t+x) + F_-(t-x), \quad (25)$$

$$\eta^x = F_+(t+x) - F_-(t-x) + f, \quad (26)$$

where F_+ and F_- are arbitrary functions of one variable, and f is a constant. The final equation to solve is equation (17):

$$2 [F'_+(t+x) + F'_-(t-x)] V + \frac{\partial V}{\partial \phi_i} (\alpha_i + \beta_{ij} \phi_j) = 0. \quad (27)$$

Since there are the free constants f , α_i and β_{ij} , and the two free functions F_\pm , the maximum rank of this system is $R = (1 + N + N(N-1)/2, 2)$.

For $D \neq 2$ we can solve equation (24) for $\partial_\sigma \eta^\sigma$ and then proceed to determine the general solution for η^μ . It is

$$\eta^\mu(x) = a^\mu + b^\mu{}_\nu x^\nu - \frac{2\gamma}{D-2} x^\mu. \quad (28)$$

Here, a^μ corresponding to coordinate translations, $b^{\mu\nu}$ is antisymmetric and corresponds to coordinate rotations and boosts, and γ gives the scaling of the coordinates. Note the similarities between this equation and equation (23). Also note that γ is the only symmetry parameter connecting the field and coordinate sectors. The maximum rank of the system is the number of free constants in the general solutions (23) and (28), which is $R = (D + D(D-1)/2 + 1 + N + N(N-1)/2)$. The remaining equation to solve for this $D \neq 2$ branch is

$$-d\gamma V + \frac{\partial V}{\partial \phi_i} (\alpha_i + \beta_{ij} \phi_j + \gamma \phi_i) = 0, \quad (29)$$

where we defined $d \equiv 2D/(D-2)$.

We have now reduced the original set of determining equations to equation (27) for $D = 2$, and equation (29) for $D \neq 2$. We have also solved for the generic form of η^μ and χ_i . Further progress can be made if one chooses a specific N and/or V . Once these are specified, one can continue to find any relationships between the symmetry parameters α_i , β_{ij} , γ , F_+ and F_- . Alternatively, one can specify these parameters — hence specify a symmetry — and look for a potential V that allows for this. In what follows we focus on a few of the simpler cases.

A. The case $D = 2$ and $N = 1$

In two dimensions with one field we have the general solution $\chi = \alpha$, which is a shift symmetry. Depending on the form of V , α may be restricted. The final determining equation (27) becomes

$$2X(t, x)V + \frac{dV}{d\phi} \alpha = 0, \quad (30)$$

where $X(t, x) = F'_+(t+x) + F'_-(t-x)$. If $X(t, x)$ is not a constant then the only solution is $V = 0$, and the symmetry rank is $R = (2, 2)$. If X is a constant (or zero) then we can solve for F_\pm and then obtain the η 's:

$$\eta^t = a^t + (b - X)x + Xt, \quad (31)$$

$$\eta^x = a^x + (b - X)t + Xx. \quad (32)$$

Since b is free we can redefine $b \rightarrow b + X$ to bring these solutions to canonical form. Then the symmetries corresponding to the constants a^t , a^x , b and X are, respectively, time translations, space translations, boosts and scaling.

For X a constant there are a few different cases to consider depending on the form of $V(\phi)$. If $V = 0$ then X does not need to be a constant, and we have already considered this case. If V is a constant then $X = 0$ but α is free, and we have a total rank $R = (4)$. For a non-trivial solution we solve the differential equation (30) for V to obtain

$$V = \lambda e^{-\frac{2X}{\alpha}\phi}. \quad (33)$$

If V is of this form then X and α are non-zero and are related given a certain choice for V . In this case the scale symmetry must work in combination with a shift of the field. The total rank for this case is $R = (4)$. Finally, for an arbitrary potential V that is non-zero, is not a constant and does not have the form of equation (33), the constants X and α must be zero and the total rank of the symmetry for this generic case is $R = (3)$.

In summary, for $D = 2$ and $N = 1$ there are the following distinct sets of symmetries:

$V = 0$: $X(t, x) = F'_+(t+x) - F'_-(t-x)$ can take any form and f and α are also free. The rank is $R = (2, 2)$.

$V = \text{const}$: $X = 0$, but $a^{t,x}$, b and α are free. The rank is $R = (4)$.

$V = \lambda e^{-m\phi}$: X and α are related by $m = 2X/\alpha$, so the scale symmetry is combined with a shift. The parameters $a^{t,x}$ and b are free. The rank is $R = (4)$.

V arbitrary: $X = \alpha = 0$, but $a^{t,x}$ and b are free. The rank is $R = (3)$.

B. The case $D \neq 2$ and $N = 1$

We now consider one scalar field in dimensions other than two. Equation (29) becomes

$$-d\gamma V + \frac{dV}{d\phi}(\alpha + \gamma\phi) = 0. \quad (34)$$

There are two parameters in the field sector, α and γ . As in the $D = 2$ case we get four distinct cases which depend on the form of the potential:

$V = 0$: α and γ free, so there exist independent shift and scale symmetries.

Rank associated with the field is $R_\chi = (2)$.

$V = \text{const}$: $\gamma = 0$ but α is free. Field rank $R_\chi = (1)$.

$V = \lambda(\phi + v)^d$: This form of V is obtained by solving the differential equation (34).

Given a specific value of v , the relationship between the shift and scale symmetry is then fixed by $v = \alpha/\gamma$. Note that if $v = 0$ then the theory has the usual scale symmetry, otherwise it is a combined shift-scale symmetry. The field rank is $R_\chi = (1)$.

V arbitrary: $\alpha = \gamma = 0$, so no shift or scale symmetry. Field rank $R_\chi = (0)$.

Recall that η^μ has solution (28), which includes coordinate shifts and rotations/boosts, along with a scaling symmetry if $\gamma \neq 0$. The total rank includes the rank from the coordinates: $R_{\text{total}} = R_\chi + R_\eta = R_\chi + (D + D(D - 1)/2)$.

C. The case $D \neq 2$ and $N = 2$

Moving on to two scalar fields, in dimensions other than two, the remaining determining equation is

$$-d\gamma V + \frac{\partial V}{\partial \phi_1}(\alpha_1 + \beta\phi_2 + \gamma\phi_1) + \frac{\partial V}{\partial \phi_2}(\alpha_2 - \beta\phi_1 + \gamma\phi_2) = 0. \quad (35)$$

Here we defined $\beta \equiv \beta_{12}$. We cannot solve this in a general way like we did in the previous cases with $N = 1$. One option is to specify a particular form for V (like a polynomial) then solve for the symmetries. Alternatively, try to find a V that yields a given symmetry. As seen in the $N = 1$ cases, it is possible to obtain non-conventional single parameter symmetries which are compositions of more familiar symmetries such as shifting and scaling. For the

$N = 2$ case there is the potential to have a relation between β and α or β and γ , yielding, respectively, combined shift-rotation and combined scale-rotation symmetries.

Actually, if we go to polar field variables then we can make some progress with equation (35). The general polar Lagrangian density is

$$\mathcal{L} = \frac{1}{2}\partial^\mu r \partial_\mu r + r^2 \frac{1}{2}\partial^\mu \theta \partial_\mu \theta - V(r, \theta) . \quad (36)$$

This has the usual Poincaré symmetry, and possibly scaling, with the general solution for η^μ given by equation (28). The polar version of the determining equation (35) is⁵

$$-d\gamma V + \frac{\partial V}{\partial r}(\alpha_1 \cos \theta + \alpha_2 \sin \theta + \gamma r) + \frac{\partial V}{\partial \theta} \left(-\alpha_1 \frac{\sin \theta}{r} + \alpha_2 \frac{\cos \theta}{r} - \beta \right) = 0 . \quad (37)$$

Note that if V does not depend on θ then β is free and there is a rotation symmetry (which manifests as a shift of the θ field). Furthermore, $\alpha_{1,2} = 0$ and the general solution for the potential is $V(r) = \lambda r^d$. Such a potential has a rotation and scale symmetry, but they act independently.

If V does depend on θ but now not on r then we find that $\alpha_{1,2} = 0$ and we are left with the differential equation

$$d\gamma V + \frac{dV}{d\theta} \beta = 0 . \quad (38)$$

The solution for the potential is $V(\theta) = \lambda \exp(-d\gamma\theta/\beta)$. If the potential has this form then γ/β is fixed and the symmetry acts by scaling x^μ and r in combination with a shift of θ . This is in effect a combined scale-rotation symmetry, or a spiral symmetry.⁶ We can also obtain other spirals. For example, with $\alpha_{1,2} = 0$, the following will work:

$$V(r, \theta) = \lambda (r^k - v e^{l\theta})^m . \quad (39)$$

The parameters λ , v and l are free, while k and m are related by $mk = d$ (for example, for $D = 4$ one can choose $k = m = 2$). The relationship between the scale and rotation symmetry is fixed by $k\gamma = l\beta$. The action of the symmetry is $r \rightarrow e^\gamma r$, $\theta \rightarrow \theta - k\gamma/l$ and $x^\mu \rightarrow e^{-d\gamma/D} x^\mu$, which is parameterised here by γ . For small l and/or θ one can Taylor expand the exponential in equation (39) to obtain a polynomial potential which is, up to

⁵ One can either transform equation (35) directly to polar form, or rederive it from scratch in the polar basis using the Lagrangian (36). The result is the same.

⁶ The spiral symmetry acts on the polar Lagrangian (36). Due to the multivalued nature of the inverse tangent function, it is difficult to define the corresponding spiral symmetry in Cartesian field variables.

higher order corrections in l and/or θ , spirally symmetric. It would be interesting to see if this result extends to a non-Abelian rotation.

V. THE ACTION VERSUS THE EQUATIONS OF MOTION

There is a distinction between the symmetries of an action and the symmetries of the corresponding set of equations of motion, the Euler-Lagrange equations. Indeed, if G is a symmetry of a given action then G is also a symmetry of the Euler-Lagrange equations, but the converse is not necessarily true.⁷ The additional symmetries present in the Euler-Lagrange equations are physically meaningful. We shall illustrate these points with an example, which also serves to illustrate the alternative way of obtaining the determining equations as described in Section II B. While this alternative method in general requires a lot more effort, it is applicable to any set of differential equations, not just those arising from a field theory. The derivation in this section also shows in more detail how one can determine the symmetry rank of a system without actually solving for the action of the symmetry, something which is important for automated solving of large systems.

Consider a single massive real scalar in 2d, whose Euler-Lagrange equation is $\ddot{\phi} - \phi'' + m^2\phi = 0$. As per Section II B, the symmetry operator is $\alpha = \eta^t \partial_t + \eta^x \partial_x + \chi \partial_\phi$. We apply the second prolongation of this operator, $\text{pr}^{(2)} \alpha$, to the equation of motion, use the equation of motion to eliminate $\ddot{\phi}$ (alternatively ϕ''), and then equate the coefficients of all independent derivatives of the field to zero. This results in 27 raw determining equations. Only 14 of these are unique (read the following as all individually equated to zero):

$$\begin{aligned}
& \eta_\phi^t; \quad \eta_{\phi\phi}^t; \quad \eta_{t\phi}^t; \quad \eta_\phi^x; \quad \eta_{\phi\phi}^x; \quad \eta_{t\phi}^x; \\
& \eta_t^t - \eta_x^x; \quad \eta_x^t - \eta_t^x; \quad \eta_{x\phi}^t - \eta_{t\phi}^x; \quad 2\eta_{t\phi}^t - \chi_{\phi\phi}; \quad 2\eta_{x\phi}^x - \chi_{\phi\phi}; \\
& \eta_{tt}^t - \eta_{xx}^t - 3m^2\phi\eta_\phi^t - 2\chi_{t\phi}; \quad \eta_{tt}^x - \eta_{xx}^x - m^2\phi\eta_\phi^x + 2\chi_{x\phi}; \\
& \chi_{tt} - \chi_{xx} + m^2\chi - m^2\phi\chi_\phi + 2m^2\eta_t^t.
\end{aligned} \tag{40}$$

The notation is that a subscript denotes differentiation with respect to that variable. The aim is to solve this set of equations for the functions $\eta^t(t, x, \phi)$, $\eta^x(t, x, \phi)$ and $\chi(t, x, \phi)$. Simple and obvious substitutions reduce this set to eight equations. Then integrability

⁷ See Section 4.2 of Olver's book [1].

conditions (basically taking derivatives and linear combinations of the above) can simplify the system further, although we end up with more equations (10 of them, but one is actually redundant):

$$\begin{aligned}
&\eta_\phi^t ; \quad \eta_\phi^x ; \quad \chi_{t\phi} ; \quad \chi_{x\phi} ; \quad \chi_{\phi\phi} ; \\
&\eta_x^t - \eta_t^x ; \quad \eta_t^t - \eta_x^x ; \quad \eta_{tt}^t - \eta_{xx}^t ; \quad \eta_{tt}^x - \eta_{xx}^x ; \\
&\chi_{tt} - \chi_{xx} + m^2\chi - m^2\phi\chi_\phi + 2m^2\phi\eta_t^t .
\end{aligned} \tag{41}$$

Taking the derivative of the last equation with respect to ϕ and using the other equations to make eliminations one finds that $m^2\eta_t^t = 0$. At this point we must create two branches of possible solutions, one when $m = 0$ and one when $m \neq 0$. Such a branch is a key step of the LPS method. In more complicated cases it leads to more interesting relationships between parameters. When automating the procedure, this is the point at which the algorithm must also branch.

A. The $m = 0$ branch

Take $m = 0$ to begin with. Instead of writing down trivial first derivative equations like $\eta_\phi^t = 0$ we shall just solve this equation and redefine the function to not depend on that particular variable. This branch then has the functions

$$\eta^t(t, x) , \quad \eta^x(t, x) , \quad \chi(t, x, \phi) , \tag{42}$$

with determining equations

$$\begin{aligned}
&\chi_{t\phi} ; \quad \chi_{x\phi} ; \quad \chi_{\phi\phi} ; \\
&\eta_x^t - \eta_t^x ; \quad \eta_t^t - \eta_x^x ; \quad \eta_{tt}^t - \eta_{xx}^t ; \quad \eta_{tt}^x - \eta_{xx}^x ; \quad \chi_{tt} - \chi_{xx} .
\end{aligned} \tag{43}$$

This is as far as one needs to go to determine the rank of the symmetry group for this branch (the above set of PDEs is in involution). The rank is determined by working out how much initial data one needs in order to fully specify a unique solution to the above system. This initial data is computed, via a well defined algorithm [2], to be the values of

$$\eta^t(0, 0) , \quad \eta^x(0, x) , \quad \eta_t^x(0, x) , \quad \chi(0, x, 0) , \quad \chi_t(0, x, 0) , \quad \chi_\phi(0, 0, 0) . \tag{44}$$

There are two constants, $\eta^t(0, 0)$ and $\chi_\phi(0, 0, 0)$, and four functions of one variable. We do not need to evaluate the coordinates and field at 0, any arbitrary value would do to fix the

solution. All that matters is how much data is needed. In this case the rank of the symmetry group is $R = (2, 4)$, with the η 's contributing $R_\eta = (1, 2)$ and χ , $R_\chi = (1, 2)$.

If all that is needed is the rank of the symmetry for a particular branch then one can stop here. But we shall solve the determining equations fully to show exactly what the symmetries of the system are and how they differ from the analysis performed with the action approach.

The general solution of the set of equations (43) is

$$\begin{aligned}\eta^t(t, x) &= F_+(t + x) + F_-(t - x) , \\ \eta^x(t, x) &= F_+(t + x) - F_-(t - x) + f , \\ \chi(t, x, \phi) &= G_+(t + x) + G_-(t - x) + g \phi(t, x) .\end{aligned}\tag{45}$$

As previously calculated via the initial data, we have two free constants, f and g , and four free functions of one variable, F_+ , F_- , G_+ and G_- . Each one of these corresponds to an independent continuous symmetry of the original Euler-Lagrange equation (with $m = 0$). f is a spatial translation and g is a scaling of ϕ (without scaling the coordinates).

For the functions G_\pm the equations describing the action of the symmetry are $\bar{t}' = 0$, $\bar{x}' = 0$ and $\bar{\phi}' = G_\pm(\bar{x} \pm \bar{t})$. The solution is $\bar{t} = t$, $\bar{x} = x$ and $\bar{\phi} = \phi + \epsilon G_\pm(t \pm x)$. Therefore, given a solution $\phi(t, x)$ of the equation of motion, one can add an arbitrary function $G_\pm(t \pm x)$ to that solution and the result is still a solution. Thus G_\pm corresponds to additivity/superposition of solutions of $\phi(t, x)$. (Note that the massless field ϕ has the general wave solution $\phi = w(t \pm x)$ with w arbitrary.) The symmetries due to the functions F_\pm are difficult to solve for in general, but include the Poincaré group and coordinate scaling. For example, choosing $F_+ = F_- = \text{constant}$ yields temporal translations, while choosing $F_\pm(t \pm x) = t \pm x$ yields a scaling symmetry.

Note that x -translations correspond to a distinct piece of initial data f , whereas t -translations do not. There is nothing important behind this asymmetry; it is simply because of the way we solved (43), choosing to solve first for η^t and then for η^x . It is important to remark that the derived parameter relationships and the computation of the rank is independent of the way in which the determining equations are reduced and solved.

B. The $m \neq 0$ branch

For the other branch with $m \neq 0$ we take the set of equations (41) and solve it for $\eta_t^t = 0$. We get the functional dependence

$$\eta^t(x), \quad \eta^x(t), \quad \chi(t, x, \phi), \quad (46)$$

and determining equations

$$\begin{aligned} \chi_{t\phi}; \quad \chi_{x\phi}; \quad \chi_{\phi\phi}; \quad \eta_{xx}^t; \quad \eta_{tt}^x; \quad \eta_x^t - \eta_t^x; \\ \chi_{tt} - \chi_{xx} + m^2\chi - m^2\phi\chi_\phi. \end{aligned} \quad (47)$$

From here the rank can be determined. The initial data consists of the values of

$$\eta^t(0), \quad \eta^x(0), \quad \eta_t^x(0), \quad \chi(0, x, 0), \quad \chi_t(0, x, 0), \quad \chi_\phi(0, 0, 0). \quad (48)$$

The rank is $R_\eta = (3, 0)$ and $R_\chi = (1, 2)$, giving a total rank of $R = (4, 2)$. We can see that the symmetry is less than the $m = 0$ case (as expected), and that the symmetry reduction is in the coordinate sector rather than the field sector.

The general solution of the determining equations (47) is

$$\begin{aligned} \eta^t(x) &= a^t + bx, \\ \eta^x(t) &= a^x + bt, \\ \chi(t, x, \phi) &= \int_{-\infty}^{+\infty} dk [H_+(k) e^{i(\omega t + kx)} + H_-(k) e^{i(\omega t - kx)}] + g\phi(t, x), \end{aligned} \quad (49)$$

where $\omega = \sqrt{k^2 + m^2}$. In the coordinate sector there is exactly the Poincaré group. In the field sector there is scaling, and the $H_\pm(k)$ correspond to additivity of backward and forward waves with wave number k .

This completes our analysis of a single scalar in 2d using the Euler-Lagrange approach. Compare with the analysis using the action approach in Section IV A: there we found rank $R = (2, 2)$ for the massless case and $R = (3, 0)$ for the massive case, compared to here with $R = (2, 4)$ and $R = (4, 2)$ respectively. The main difference is that the equations of motion are invariant under addition of solutions, whereas the action is not. An important question, which we shall not attempt to answer, is whether the larger class of symmetries obtained from the Euler-Lagrange equations contains anything interesting and/or important when analysing large systems. From a pragmatic point of view the action approach is much

simpler (it yields fewer determining equations) and this approach would be preferred for large systems.⁸

VI. REMARKS ON THE LPS METHOD

Of great interest is the fact that the LPS method provides an exhaustive list of relationships between parameters such that an enhanced symmetry is obtained. In the reduction and solving of the determining equations one invariably comes across equations where different solutions are obtained (and hence different symmetries) when parameters of the model take special values, or when unspecified functions like the potential take a different functional form.⁹ The converse is also true, because if special values or combinations of parameters, or special forms of unspecified functions, lead to different symmetries then the LPS method must necessarily distinguish these scenarios. In the previous sections we have demonstrated this for some simple systems. For more complex cases one can obtain much more complicated relationships among the parameters.

In the examples so far we have only considered spin-0 particles, and the LPS method has been developed to handle only a collection of real fields. But in fact any spin representation, or even particles that do not respect Lorentz symmetry, can be written in terms of real fields, and, consequently, any action can be expanded in terms of its real components. The LPS method, and in particular equation (5), is therefore general enough for the purposes of model building. For example, in 4d, gauge fields and Weyl fermions have four real components. From this point of view an action is just a bunch of real fields with certain terms being derivative interactions and other terms having the usual Yukawa form. The distinguishing feature between scalars, gauge fields and fermions is the structure of the self-coupling derivative (kinetic) terms. For a fermion such derivative terms are just right to get a spin- $\frac{1}{2}$ representation of the Lorentz group. Any field theory is then just a set of interacting real fields, and the symmetries of the theory act on this set, and on the coordinates. This

⁸ We could also consider a third class of symmetries, those of the Lagrangian density itself (not the action). This is probably not very interesting once we have the symmetries of the action and/or Euler-Lagrangian equations, but it is easy to do. We just take equation (5) and drop the first term. The resulting equation can give the determining equations for the pure Lagrangian (without the volume element).

⁹ Unspecified functions can only be functions of x and ϕ , not of derivatives of ϕ , since then one cannot obtain the determining equations in explicit form.

is a very naïve view, but with this naïvety comes freedom from bias and allows one to systematically classify and study the properties of a given model. We return to this point in Section VII.

The LPS method works for all continuous symmetries that depend on the coordinates and fields (but not derivatives of the fields). This includes local gauge symmetries [5, 6] as well as local coordinate diffeomorphisms [6]. The extension to supersymmetry is possible and requires the introduction of anti-commuting coordinates [8]. The LPS method works also for non-linear symmetries. As an example, consider the field (no coordinate) symmetries of $\mathcal{L} = \phi^m (\partial^\mu \phi \partial_\mu \phi)^n$, where m and $n \neq 0$ are constant exponents. The solution to the determining equations is $\chi = a\phi^{-m/2n}$ with constant a . The corresponding non-linear symmetry acts by $\phi \rightarrow (\phi^p + pa\epsilon)^{1/p}$, with $p = 1 + m/2n$.

Section IV B gave a simple example where a spontaneously broken scale symmetry was found, with potential $V = \lambda(\phi + v)^d$. The corresponding unbroken potential is $V = \lambda\phi^d$, which is manifestly scale invariant. The potential written in the broken phase still has the same symmetry rank, but now the particular scale symmetry is implemented by a combined shift-scale. The symmetry acts by first shifting the field to the unbroken phase, then scaling it, then shifting it back to the broken phase. This is a generic feature of spontaneously broken symmetries. A Lagrangian that is symmetric under a certain group, which is then expanded around some vacuum state, will still contain essentially the same symmetry, just implemented in a slightly different way. At the level of the action, one cannot break a symmetry by a simple field redefinition (it is the vacuum state, hence a solution of the equations of motion, that breaks the symmetry). This means that the LPS method will always be able to find a symmetry of an action, even if the action is written in the broken phase.

In Section IV we analysed N interacting spin-0 fields, and proved that the most general symmetries of such a system are given by equations (23) and (28) (for dimensions other than 2d). The LPS method thus gives a complementary approach to the proofs regarding all symmetries of the S-matrix [11, 12]. The method at hand also gives a straightforward way to compute the precise symmetries for a given model. Furthermore, as pointed out above, it is applicable to spontaneously broken symmetries.

Looking beyond point symmetries, one has contact symmetries, which depend also on the first derivative of the field, and generalised symmetries (sometimes called Lie-Bäcklund),

which allow η^μ and χ_i to depend on arbitrary derivatives of ϕ_i . These can be handled with suitable extensions of the LPS method, although solving the determining equations becomes more involved. There are also discrete symmetries, which, apart from those that are subsets of a continuous group, are not covered by the methods outlined in this paper. One can easily show that continuous and discrete symmetries act independently: if $g \rightarrow C(D(g))$ is a symmetry then so is $g \rightarrow C(g)$ and $g \rightarrow D(g)$, for C a continuous, and D a discrete symmetry. Thus, if we find all the continuous symmetries, and then all discrete symmetries separately, then we have found all of the symmetries of the system. Ref. [13] discusses a method to systematically find discrete point symmetries, which should be applicable to model building. See also Ref. [14] for a systematic study of discrete symmetries in the context of model building.

VII. AUTOMATION, AND A CATALOGUE OF ALL FIELD THEORIES

Applying the LPS method to a large, complex system can lead to an unmanageable set of determining equations. Fortunately, the procedure can be cast as a well defined algorithm that completes in finite time, at least up to finding parameter relationships and the rank of the symmetry. It is therefore feasible to construct a computer program which takes in a Lagrangian and returns a list of branches, where each branch corresponds to a different set of symmetries and consists of the associated rank and parameter relationships. In addition the branch can contain the reduced determining equations, which can be further solved if needed. We shall outline how such a program can be constructed, and then present a few ideas on how it can be put to use.

First of all the program must compute the determining equations. This is straightforward using standard computer algebra, although for systems with a large number of degrees of freedom one must be careful to use symbolic algebra algorithms that have low order complexity in the number of terms.

Reducing the determining equations to standard form is the difficult part. Let us first discuss how this works for the simplified case where the determining equations are reduced to algebraic form, as happened in the example with two massive scalars in the derivation of equation (14). From that equation one obtains three independent constraints which can be

written in matrix form as

$$\begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_1^2 - m_2^2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \beta \end{pmatrix} = 0. \quad (50)$$

Any set of algebraic determining equations can be written this way, with the matrix containing the free parameters of the theory, and the column vector the symmetry parameters. We are interested in this matrix's null space, which can be different for special values of the model parameters. In the above example, if the masses are not special then the null space is trivial, $(\alpha_1, \alpha_2, \beta) = 0$ and there are no symmetries. But, if $m_1^2 = 0$, $m_2^2 = 0$ or $m_1^2 = m_2^2$, then one of the rows of the matrix is eliminated and the dimension of the null space is at least one, meaning there is at least one independently acting symmetry. For large systems the matrix can be large and contain off diagonal entries, and there can be many cases to consider. The problem can be solved systematically by implementing a Gaussian elimination algorithm that takes into account the possibility of a leading entry being zero or non-zero, and producing a new branch at such a point.

For the general case the determining equations are linear partial differential equations in η^μ and χ_i , and one reduces the system to “diagonal” form using a generalised version of Gaussian elimination. For this to make sense, one first defines a strict ordering of the variables η^μ and χ_i and their derivatives, with higher order derivatives coming first. The terms in each determining equation are then sorted using this ordering. In principle this linear system can then be written as a matrix of coefficients (which can depend on x^μ and ϕ_i) operating on a vector of all possible derivatives of η^μ and χ_i . Generalised Gaussian elimination can then proceed, with additional operations such as differentiation of a matrix row. In practice the matrix is very sparse and it is easier to implement the rules of reduction directly on the determining equations.

It is during the equivalent of column elimination that branching of the solution can occur. Schematically, the leading order terms in a pair of determining equations looks like

$$\begin{aligned} c_1 \partial_i f + X_1(f) &= 0, \\ c_2 \partial_{i+j} f + X_2(f) &= 0. \end{aligned} \quad (51)$$

Since the derivative ∂_i is contained within ∂_{i+j} we can use the first equation to eliminate the $c_2 \partial_{i+j} f$ term in the second equation. But this can be done only if the leading coefficient c_1

is non-zero. Since c_1 in general depends on free parameters, or free functions, of the original model, c_1 being zero or non-zero defines a particular branch point in the solution, and a particular relationship between parameters that may lead to a different set of symmetries. Each branch is reduced until no more eliminations can be done, at which point the system is in involution and includes all of its integrability conditions. In this way the determining equations are systematically reduced, with all branching accounted for. See Reid [2, 3] for a more detailed description of this algorithm, and also for an algorithm which computes the initial data, the rank, of the reduced set of determining equations.

Let us now assume that we have a program which, given a Lagrangian, can tell us in a reasonable amount of time all the possible branches and their corresponding parameter constraints and symmetry rank (and possibly also the symmetry group). Now, using the observation that any model can be written in terms of its real components, we can start to make a comprehensive catalogue of all possible theories, at least within some limit.

Such a catalogue will be ordered on the number of real degrees of freedom N . Given this number, we literally just write down the most general action, with general derivative couplings and Yukawa couplings between all N degrees of freedom. Feeding the action into our program we obtain a large but finite list of all the possible relationships among parameters and all the symmetries. This will include relationships amongst the coefficients of the derivative terms in order to get particles of spin-0, spin- $\frac{1}{2}$ and so on. For a given N , given number of dimensions and given highest-order coupling, this is a well defined and finite procedure. Any model one can think of will be in this catalogue.

For instance, in 4d with $N = 4$ one will find electromagnetism, $N = 6$ contains scalar QED, and $N = 8$ has QED with one Weyl fermion. With $N = 10$ one will find general relativity, among many other theories. It may seem that one will only “find” general relativity because one had prior knowledge of what to look for, but it is arguable that the symmetry rank of general relativity is so much larger than others in the $N = 10$ class that it would stand out from the rest of the branches. If true, simply by sorting the branches on their rank would allow one to literally “discover” general coordinate invariance.

As an explicit example, with $N = 1$ in 2d with up to bi-linear terms we would write

$$\mathcal{S} = \int dt dx \left[c_0 + c_1\phi + c_2\dot{\phi} + c_3\phi' + c_4\phi^2 + c_5\phi\dot{\phi} + c_6\phi\phi' + c_7\dot{\phi}^2 + c_8\dot{\phi}\phi' + c_9\phi'^2 \right], \quad (52)$$

where the c_i are constant parameters of the theory. Feeding this into our program would

give us all the relationships among the c_i along with the corresponding symmetries. It would include the case $c_0 = c_1 = c_2 = c_3 = c_5 = c_6 = c_8 = 0$, $c_7 = -c_9$ which is a massive spin-0 field in 2d.

The obvious drawback of all this is the computational limits in time and storage that will be hit for moderately sized systems. The reason is that the number of terms in a general action grows combinatorically with the number of fields, and even more so with the number of coordinates. For D coordinates, N fields and a maximum of F factors in each term, there are

$$T = \frac{(F + N + DN)!}{F!(N + DN)!} = \frac{(F + N + DN) \cdots (1 + N + DN)}{F!} \quad (53)$$

distinct terms. For $D = F = 2$ we have $T = (9N^2 + 9N + 2)/2$ which behaves asymptotically like N^2 . For $D = F = 4$ (4d with renormalisable terms) we obtain $T \sim 26N^4$ for large N , and for $N = 10$ there are about 3×10^5 terms (and this many free parameters), which may be manageable by a computer. At least for $N = 4$ it will most likely be manageable with $T \sim 10^4$, allowing for a gauge field or a Weyl fermion.

To ease the combinatorical problem we can restrict the derivative terms to a known spin structure. The class of models to be considered is then designated by the number of spin-0 fields, number of spin- $\frac{1}{2}$ fields (Weyl) and so on, and the number of parameters in the initial action. For each model we obtain a list of all the branches of possible sets of symmetries, with the number of remaining free parameters, and the rank of the symmetry group. The classification label for a model \mathcal{M} might look something like

$$\mathcal{M} = (N_{\text{spin-0}}, N_{\text{spin-}\frac{1}{2}}, N_{\text{spin-1}}, N_{\text{param}}) \rightarrow \{(N_{\text{free param}}, R_\eta, R_\chi)\} , \quad (54)$$

where N_X is the number of X and R is the symmetry rank. For a single spin-0 particle in 2d with mass parameter m we would get (the $m = 0$ branch is listed first)

$$\mathcal{M} = (1, 0, 0, 1) \rightarrow \{[0, (1, 2), (1, 0)], [1, (3, 0), (0, 0)]\} \quad (\text{using the action}) \quad (55)$$

$$\mathcal{M} = (1, 0, 0, 1) \rightarrow \{[0, (1, 2), (1, 2)], [1, (3, 0), (1, 2)]\} \quad (\text{using Euler-Lagrange}) \quad (56)$$

For large N there are more and more ways of splitting the real fields into specific spin representations; in 4d for N fields there are $\frac{1}{2}(M + 1)(M + 2)$ different splits, where $M = \text{floor}(N/4)$. This grows only mildly as $N^2/32$. For example, with $N = 5$ there are three splittings: either five spin-0 fields, one spin-0 and one spin- $\frac{1}{2}$, or one spin-0 and one spin-1. If we add up all the possible \mathcal{M} 's for a maximum of $N = 20$ real fields we obtain 160 models,

where each model is well defined and contains all possible interactions that respect the given Lorentz structure of the kinetic terms. $N = 20$ may be manageable on a computer, and is just enough to include $SU(2) \times U(1)$ gauge theory with a complex Higgs doublet.

The catalogue does not need to be restricted to 4d with renormalisable couplings. It really is only limited by one's imagination and available processing power. The LPS method can handle extra dimensions, non-standard kinetic terms, higher-order operators, supersymmetry, and anything else that can be written down in an action or with equations of motion.

Although probably not feasible in the near future, we would ultimately like to construct a catalogue that includes all 4d theories up to the standard model and beyond, requiring N in the hundreds. If the theory beyond the standard model can be described by an action in 4d then the catalogue would contain this theory. From this point of view the putative new theory is part of a *finite* set (for large but finite N). Using the LPS method, this is a finite set that we know how to compute. If Nature has chosen something highly symmetric then one has the chance of finding the theory beyond the standard model by sorting the catalogue on the rank of the symmetry of the branch of each model, and looking at those with large rank. Although these ideas are highly speculative, they provide an alternative perspective on constructing physics models beyond the standard model.

What we have described in this section is our new approach to model building. We are no longer thinking in terms of unification of gauge groups, what larger groups contain the standard model, or what matter representations we should choose. We forget all that. *We take the naïve perspective of an action as a bunch of real fields with derivatives, couplings and parameters. We systematically break down a model into these rudimentary components, and then use the LPS method to build the model back up in a systematic way, finding all possible re-constructions, and all interesting parameter relationships.*

VIII. THE STANDARD MODEL

Perhaps the most obvious thing to do first with the LPS method is to find all the continuous symmetries of the standard model. Nature has allowed us to discern this model and we should be absolutely certain that we are not missing anything. It may be that there is something subtle in the standard model that is not obvious without a systematic explo-

ration. But it is most likely that a search reveals only what we already know. Even so, we would have then proven the following: as is, the standard model has no new symmetries and the parameters are pure inputs with no meaningful relationships. In order to simplify the standard model we must extend it.

Consider, in a schematic way, the general structure of the standard model

$$\mathcal{L}_{\text{SM}} \sim (\partial\phi)^2 + \phi^2\partial\phi + \phi^2 + \phi^4 + \psi\partial\psi + \phi\psi^2, \quad (57)$$

where ϕ is a real field with mass dimension M^1 and can be a scalar or gauge field component, and ψ is also a real field but with dimension $M^{3/2}$ so represents a fermionic component. In comparison with (16), the second and fifth terms here are new. These terms lead to much more involved structure in the determining equations, and allow for non-Abelian gauge symmetries and spin- $\frac{1}{2}$ representations of the Lorentz group. It would be interesting to see if such terms also allow for new non-compact symmetries relating coordinates and fields.

Written out as a bunch of real fields in the form of equation (57), the standard model with right-handed neutrinos has $N = 244$ real degrees of freedom,¹⁰ making it a formidable beast indeed. Using the action approach, the number of terms in the master determining equation (5) goes like $N \times (\text{terms in } \mathcal{L}) + N^2$. With approximately N^2 terms in \mathcal{L} , this gives of the order of 10^7 total terms for $N = 244$. In this case, the maximum number of determining equations (set by the number of independent derivatives of fields with up to three factors, like $\partial\phi_1\partial\phi_2\partial\phi_3$) is 2.5×10^6 . In contrast, starting with the Euler-Lagrange equations gives orders of magnitude more complexity. The twice prolonged operator $\text{pr}^{(2)}\alpha$ alone has about 10^9 terms in it for $N = 244$, and the operator must be applied to 244 equations. The action approach seems favourable, and there are further simplifications and tricks we can apply to make it more manageable.

It would be desirable to generalise the analysis of Section IV by adding to the Lagrangian (16) the extra terms necessary to encompass all the terms in the standard model. In this way it may be possible to show precisely the symmetries allowed by the generic Lagrangian (57), and further reduce the determining equations to algebraic form. This then makes the final simplification stage pure Gaussian elimination, as discussed in Section VII.

¹⁰ We have: gauge = 4 real components \times (1 hyp + 3 weak + 8 strong) = 48, leptons = 8 real components \times 3 gens \times (ν + e) = 48, quarks = 8 real components \times 3 gens \times 3 cols \times (u + d) = 144, and Higgs = 2 real components \times weak-doublet = 4. Total = 244.

The number of fields can be reduced by turning off certain parts of the standard model. Eliminating the colour and quark sector leaves only $N = 68$ real fields. Here one could study electroweak symmetry breaking, lepton family symmetries and neutrino masses and mixing. A single generation of only the leptons gives $N = 36$, and a single generation with the colour sector included has $N = 116$.

Unfortunately, we can not use our knowledge of the known symmetries of the standard model to simplify the analysis. Additional symmetries mean a more general form of η^μ and/or χ_i , so if one makes an ansatz for these functions based on known symmetries, then one has immediately excluded the possibility of finding anything new.

Reducing the determining equations to standard form and obtaining a set of branches is the most interesting part, and also the most difficult since the number of branches may become unmanageably large. We can decrease the number of branches that are taken by using our knowledge of the values of the parameters in the standard model. At each branch point the numerical value of the coefficient, which is a function of the parameters, is checked for zero within the uncertainty of the experimental value of the parameters. Only if it is zero within the range is a branch taken. If this is manageable, then it should be possible to leave a couple of parameters completely free and always branch when a coefficient is dependent on them. This technique will also be useful for adding new degrees of freedom to the standard model with unknown couplings. One could also linearise all parameters around their known value, which is easier than the non-linearised case because one only needs to solve a linear equation at each branch point.

When checking at a branch point if a combination of parameters is zero or not, one needs to take into account the fact that actual parameter values run and depend on the energy scale. This is not difficult, but needs to be considered. For example, one could run the energy scale upward in small steps, and at each step look for new symmetries, for new branchings that occur due to the coefficient of a leading term being zero at a particular energy. The existence of a new symmetry of the standard model manifesting at some particular energy scale due to the parameters unifying is exactly what we expect to happen with unification of the gauge groups.

The Higgs has not been measured, so to be pedantic one would first look for the symmetries of the standard model without the Higgs sector. Starting with the theory in the electroweak broken phase with bare mass terms, an LPS search would yield at least Lorentz,

and $SU(3)$ and $U(1)$ gauge symmetries. Moving on from there, new real degrees of freedom would be added, along with all their allowed interactions, and the LPS search repeated to look for additional symmetries. The important thing to realise is that adding four real degrees of freedom will reduce the number of parameters by one and introduce a new gauge symmetry, $SU(2)$ (the Higgs mechanism). Since the LPS method can handle symmetries written in the broken phase, it should be able to re-discover the Higgs mechanism in this way. Going from 240 to 244 degrees of freedom reduces the parameters by one and increases the total rank of the symmetries. An interesting question is whether there exists a parameter relationship which is not the standard Higgs mechanism. The LPS method can give a definitive answer to this, at least in the regime of adding only a small number of new real fields.

Assume we have shown this, that we can systematically find the correct degrees of freedom and interactions that allows an increase in the symmetry and a reduction in the number of parameters. Then we can ask the following question: is it possible to make the standard model more predictive (reduce the number of parameters) within the framework of a 4d Lagrangian with operators? We can start to answer this question by doing a search of the standard model plus new degrees of freedom. If the LPS method can find the Higgs mechanism, then it should be able to find the next symmetry group beyond that and the associated parameter relationship(s). If, for a large number of new degrees of freedom, the LPS search comes up empty handed then we conclude that the standard model cannot be made more predictive using real degrees of freedom in 4d. This would point to, for example, the necessity of extra dimensions and/or supersymmetry, both of which can be analysed by a more extensive LPS search. In fact, any model which can be written as an action or a set of equations of motion is amenable to the LPS technique of systematically finding symmetries and relationships among parameters. The only limitation is computing power and interpreting the output.

IX. CONCLUSIONS

We have described a method for systematically and exhaustively searching for all continuous symmetries of a model. The model of interest can be described by an action, Lagrangian density, equations of motion, or any set of coupled PDEs. Using the action approach, the

master determining equation (5) provides a counterpart to the Euler-Lagrange equations. It essentially extracts a linearised version of a theory whose solutions are the symmetries. The LPS method also provides a systematic way to find the solutions of these equations, or at least their rank. Along with a list of all symmetries, the method will also give a systematic list of all interesting relationships between free parameters (or even free functions), where interesting means that a different set of symmetries can be obtained.

The LPS method itself is not new. What is new is the application to model building. Of great interest is to use the method to find all the symmetries of the standard model, and then go beyond to find the simplest extensions which yield a reduction in the number of parameters. One could also construct a comprehensive catalogue of all possible 4d field theories for low numbers of real degrees of freedom.

In the 35 or so years since the standard model was written down, there has not been one model which is more predictive and reduces the number of parameters (disregarding the neutrino sector). But there have been countless attempts at this. Attempts to intuitively guess a bigger symmetry group and then from it derive the standard model and its couplings. The method described in this paper gives a new approach to model building by providing a much more systematic way to search for extensions of, for example, the standard model. Given a particular extension of degrees of freedom and couplings, the method allows one to find all possible symmetries and *derive* constraints among parameters, removing part of the guess work in model building.

Acknowledgments

I would like to thank M. Postma, B. Schellekens, R. de Adelhart Toorop and S. Mooij for stimulating discussions and reading a past manuscript. I would also like to thank R.R. Volkas, R. Davies and K.L. McDonald for very helpful comments on a previous version of this paper. This research was supported by the Netherlands Foundation for Fundamental Research of Matter (FOM) and the Netherlands Organisation for Scientific Research

(NWO).

-
- [1] P. J. Olver, *Applications of Lie Groups to Differential Equations*. New York: Springer-Verlag, 1986.
 - [2] G. J. Reid, *A triangularization algorithm which determines the Lie symmetry algebra of any system of PDEs*, *Journal of Physics A: Mathematical and General* **23** (1990), no. 17 L853.
 - [3] G. J. Reid, *Algorithms for Reducing a System of PDEs to Standard Form, Determining the Dimension of its Solution Space and Calculating its Taylor Series Solution*, *Eur. J. of Appl. Math.* **2** (1991) 293–318.
 - [4] G. Reid, I. Lisle, A. Boulton, and A. D. Wittkopf, *Algorithmic determination of commutation relations for Lie symmetry algebras of PDEs*, in *Proc. ISSAC '92*, ACM Press, Berkeley, 1992.
 - [5] W. Hereman, L. Marchildon, and A. M. Grundland, *Lie point symmetries of classical field theories*, in *Proceedings of the XIX International Colloquium, Salamanca, Spain, June 29-July 4, 1992, Anales de Física. Monografías, Group Theoretical Methods in Physics, vol. 1* (M. A. del Olmo, M. Santander, and J. M. Guilarte, eds.), pp. 402–405, Real Sociedad Española de Física, 1993.
 - [6] L. Marchildon, *Lie symmetries of Yang-Mills equations*, *J. Group Theor. Phys.* **3** (1995) 115–130, [math-ph/0307028].
 - [7] L. Marchildon, *Lie symmetries of Einstein's vacuum equations in N-dimensions*, *J. Nonlin. Math. Phys.* **5** (1998) 68–81, [gr-qc/9701044].
 - [8] A. Grundland, A. Hariton, and L. Snobl, *Invariant solutions of the supersymmetric sine-Gordon equation*, *J. Phys. A* **A42** (2009) 335203, [arXiv:0812.3862].
 - [9] W. Hereman, *Symbolic Software for Lie Symmetry Analysis*, in *CRC Handbook of Lie Group Analysis of Differential Equations, Volume 3: New Trends in Theoretical Developments and Computational Methods, Chapter 13* (N. Ibragimov, ed.), pp. 367–413, CRC Press, Boca Raton, Florida, 1996.
 - [10] D. F. M. Torres, *Proper Extensions of Noether's Symmetry Theorem for Nonsmooth Extremals of the Calculus of Variations*, *Communications on Pure and Applied Analysis* **3** (September, 2004) 491–500.

- [11] S. R. Coleman and J. Mandula, *All possible symmetries of the S matrix*, *Phys.Rev.* **159** (1967) 1251–1256.
- [12] R. Haag, J. T. Lopuszanski, and M. Sohnius, *All Possible Generators of Supersymmetries of the s Matrix*, *Nucl.Phys.* **B88** (1975) 257.
- [13] P. Hydon, *How to Construct the Discrete Symmetries of Partial Differential Equations*, *Eur. J. of Appl. Math.* **11** (2000) 515–527.
- [14] C. I. Low and R. R. Volkas, *Tri-bimaximal mixing, discrete family symmetries, and a conjecture connecting the quark and lepton mixing matrices*, *Phys.Rev.* **D68** (2003) 033007, [hep-ph/0305243].